**Partial Orders**

**Partial ordering relation** is a binary relation $\leq$that is reflexive, antisymmetric, and transitive, that is, the following properties hold for all $x,y,z$:

* $x \leq x$
* $x \leq y \land y \leq x \rightarrow x=y$
* $x \leq y \land y \leq z \rightarrow x \leq z$

If $A$is a set and $\le$a binary relation on $A$, we call the pair $(A,\le)$a **partial order**.

Given a partial ordering relation $\le$, the corresponding **strict ordering relation** $x < y$is defined by $x \le y \land x \neq y$and can be viewed as a shorthand for this conjunction.

We can view partial order $(A,r)$as a first-order interpretation $I=(A,\alpha)$of language ${\cal L}=\{\le\}$where $\alpha({\le})=r$.

**Example Partial Orders**

Orders on integers, rationals, reals are all special cases of partial orders called linear orders.

Given a set $U$, let $A$be any set of subsets of $U$, that is $A \subseteq 2^U$. Then $(A,\subseteq)$is a partial order.

**Example:** Let $U = \{ 1,2,3\}$and let $A = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{2,3\} \}$. Then $(A, \le)$is a partial order. We can draw it as a Hasse diagram.

**Hasse diagram**

[Hasse diagram](http://www.google.com/search?q=Hasse%20diagram&btnI=lucky) presents the relation as a directed graph in a plane, such that

* the direction of edge is given by which nodes is drawn above
* transitive and reflexive edges are not represented (they can be derived)

**Extreme Elements in Partial Orders**

Given a partial order $(A,\le)$and a set $S \subseteq A$, we call an element $a \in A$

* **upper bound** of $S$if for all $a' \in S$we have $a' \le a$
* **lower bound** of $S$if for all $a' \in S$we have $a \le a'$
* **minimal element** of $S$if $a \in S$and there is no element $a' \in S$such that $a' < a$
* **maximal element** of $S$if $a \in S$and there is no element $a' \in S$such that $a < a'$
* **greatest element** of $S$if $a \in S$and for all $a' \in S$we have $a' \le a$
* **least element** of $S$if $a \in S$and for all $a' \in S$we have $a \le a'$
* **least upper bound** (lub, supremum, join, $\sqcup$) of $S$if $a$is the least element in the set of all upper bounds of $S$
* **greatest lower bound** (glb, infimum, meet, $\sqcap$) of $S$if $a$is the greatest element in the set of all lower bounds of $S$

Taking $S=A$we obtain minimal, maximal, greatest, least elements for the entire partial order.

Duality minimal/maximal, least/greatest, supremum/infimum

Notes

* minimal element need not exist: $(0,1)$interval of rationals
* there may be multiple minimal elements: $\{\{a\},\{b\},\{a,b\}\}$
* if minimal element exists, it need not be least: above example
* there are no two distinct least elements for the same set
* least element is always glb and minimal
* if glb belongs to the set, then it is always least and minimal
* for relation $\subseteq$on sets, $glb$is intersection, $lub$is union (not all families of sets are closed under $\cap$, $\cup$)

**Monotonic functions**

Given two partial orders $(C,\le)$and $(A,\sqsubseteq)$, we call a function $\alpha : C \to A$*monotonic* iff for all $x, y \in C$,   
\begin{equation*}
  x \leq y\ \rightarrow\ \alpha (x) \sqsubseteq \alpha (y)
\end{equation*}